On maximal S-free sets and the Helly number for the family of S-convex sets

Gennadiy Averkov*

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Abstract

Let S be a subset of $\mathbb{R}^d$. A subset K of $\mathbb{R}^d$ is said to be S-free if K is closed, convex and the interior of K is disjoint with S. An S-free set K is said to be maximal if K is not properly contained in another S-free set. We present a condition on S which guarantees that every maximal S-free set is a polyhedron with at most f facets, where the bound f depends only on S. This condition on S is formulated in terms of the Helly number for the family of S-convex sets. The presented result yields corollaries related to the cutting-plane theory from integer and mixed-integer optimization.

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1 Introduction

Let $d \in \mathbb{N}$ and $S \subseteq \mathbb{R}^d$. A subset K of $\mathbb{R}^d$ is said to be *S-free* if K is closed, convex and the interior of K is disjoint with S. An S-free set K in $\mathbb{R}^d$ is said to be *maximal* if there exists no S-free set properly containing K. Dey and Morán [7] have recently studied maximal S-free sets in the case that S is the intersection of $\mathbb{Z}^d$ and a convex set. In particular, in [7] the following result was obtained.

Theorem 1. (Dey & Morán, [7, Theorem 3.4]). *Let $S = \mathbb{Z}^d \cap C$, where $C \subseteq \mathbb{R}^d$ is convex. Then every d-dimensional maximal S-free set is a polyhedron with at most 2^d facets.*

In the case $S = \mathbb{Z}^d$ Theorem 1 was formulated by Lovász [11, Theorem 3.4] (a proof can be found in [4, §2.2]). Various special cases of Theorem 1 were considered and used in [4, 5, 8, 10]. This note presents a theorem (Theorem 4), which implies Theorem 1 and also yields the following mixed-integer analog of Theorem 1.

Theorem 2. *Let $d, n \in \mathbb{N}$ and $S = (\mathbb{Z}^d \times \mathbb{R}^n) \cap C$, where $C \subseteq \mathbb{R}^d \times \mathbb{R}^n$ is convex. Then every d-dimensional maximal S-free set is a polyhedron with at most 2^d facets.*

*Institute for Mathematical Optimization, Faculty of Mathematics, University of Magdeburg, 39106 Magdeburg, Germany; email: averkov@math.uni-magdeburg.de

We remark that maximal S-free sets of dimension less than d can be characterized in rather simple terms: a subset K of $\mathbb{R}^d$ is maximal S-free and of dimension less than d if and only if K is a hyperplane and in both open subspaces determined by K one can find points of S lying arbitrarily close to K. Hence in what follows our considerations are restricted to the case of d-dimensional maximal S-free sets.

A natural question in the context of the cutting-plane theory (from integer and mixed-integer programming) is whether for a given S one can find $f \geq 0$ such that every maximal S-free set is a polyhedron with at most f facets. The motivation provided by the cutting-plane theory is based on the fact that maximal S-free sets for $S := \mathbb{Z}^d \cap C$ (where C is convex) can be used as 'cutting objects' for generation of intersection cuts (see, for example, [1, 3, 10]). It is thus desirable to have an upper bound on the combinatorial complexity of such cutting objects.

Our question on the existence of an upper bound f can be expressed in terms of a parameter $f(S)$, which we introduce as follows. If, for a given S, there exist maximal S-free sets which are not polyhedra or if there exist d-dimensional maximal S-free polyhedra with arbitrarily large number of facets we let $f(S) := +\infty$. If there exist no d-dimensional maximal S-free sets (e.g., for $S = \mathbb{R}^d$) we let $f(S) := -\infty$. In the remaining cases the set of d-dimensional maximal S-free sets is nonempty and consists of polyhedra whose number of facets is bounded in terms of S; in such cases we denote by $f(S)$ the largest possible number of facets in a d-dimensional maximal S-free polyhedron. Thus, we ask for conditions on S which ensure $f(S) < +\infty$. The main message of this note is that there exists a strong relation between $f(S)$ and the Helly number associated to the family of S-convex sets.

Definition 3. Let $\mathcal{F}$ be a nonempty family of sets with $\mathcal{F} \neq \{\emptyset\}$. Then the *Helly number* $h(\mathcal{F})$ of $\mathcal{F}$ is defined to be the minimal $h \in \mathbb{N}$ such that the following implication holds: If $\mathcal{X}$ is an arbitrary finite subfamily of $\mathcal{F}$ such that $\mathcal{X}$ contains at least h sets and every h-element subfamily of $\mathcal{X}$ has nonempty intersection, then also $\mathcal{X}$ has nonempty intersection. If no $h \in \mathbb{N}$ as above exists, we let $h(\mathcal{F}) := +\infty$. We also define the Helly number of $\{\emptyset\}$ by $h(\{\emptyset\}) := 0$.

A subset A of $\mathbb{R}^d$ is called *S-convex* if $A = S \cap C$ for some convex subset C of $\mathbb{R}^d$. The notion of S-convexity is reduced to the standard notion of convexity for $S = \mathbb{R}^d$ and to the notion of lattice-convexity for $S = \mathbb{Z}^d$. Let $h(S)$ denote the Helly number for the family of all S-convex sets. That is

$$h(S) := h\left(\left\{S \cap C \,:\, C \text{ is a convex subset of } \mathbb{R}^d\right\}\right).$$

The equality

$$h(\mathbb{R}^d) = d + 1 \tag{1}$$

represents the classical theorem of Helly (see, for example, [13, Theorem 1.1.6]). Doignon [9, (4.2)] proved the equality

$$h(\mathbb{Z}^d) = 2^d, \tag{2}$$

which is the analog of Helly's theorem for $\mathbb{Z}^d$-convex sets. See also [14, §16.5] for interpretation of (2) in terms of integer optimization. The result of Doignon and its special cases have often been rediscovered (see [6, 12, 15]). Based on (1) and (2) the authors of [2] showed

$$h(\mathbb{Z}^d \times \mathbb{R}^n) = (n + 1)2^d \tag{3}$$

for all $d, n \in \mathbb{N}$. Equality (3) is the mixed-integer analog of Helly's theorem.

Now we are ready to formulate our main result.

Theorem 4. *Let $S \subseteq \mathbb{R}^d$. Then $f(S) \le h(S)$.*

As a consequence of Theorem 4 and (2) we obtain

Theorem 5. *Let $d, n \in \mathbb{N}$. Let $A \subseteq \mathbb{R}^d$ and $B \subseteq \mathbb{R}^d \times \mathbb{R}^n$ be convex. Then*

$$f(A \cap \mathbb{Z}^d) \le 2^d, \tag{4}$$

$$f(\mathbb{Z}^d) = 2^d, \tag{5}$$

$$f(B \cap (\mathbb{Z}^d \times \mathbb{R}^n)) \le 2^d, \tag{6}$$

$$f(\mathbb{Z}^d \times \mathbb{R}^n) = 2^d. \tag{7}$$

Comparing (5), (7) with (2), (3) we see that, for different choices of S, in Theorem 4 one can have the equality $f(S) = h(S)$ as well as the strict inequality $f(S) < h(S)$.

Inequalities (4) and (6) represent the assertions of Theorems 1 and 2, respectively. The authors of [7] indicate that their proof of Theorem 1 is quite technical (see [7, p. 382, remark after Proposition 3.3]). In contrast to this, our arguments lead to a shorter and less technical proof of Theorem 1.

2 Proofs

We use standard terminology from the theory of polyhedra (see, for example, [14, Part III]). For $n \in \mathbb{N}$ let $[n] := \{1, \ldots, n\}$. The standard scalar product of $\mathbb{R}^d$ is denoted by $\langle \cdot \, , \, \cdot \rangle$. By int we denote the interior with respect to the Euclidean topology of $\mathbb{R}^d$.

Lemma 6. *Let $S \subseteq \mathbb{R}^d$ and $f \in \mathbb{N}$. Assume that every d-dimensional S-free rational polyhedron P is contained in an S-free polyhedron Q with at most f facets. Then every d-dimensional maximal S-free set is a polyhedron with at most f facets.*

Proof. Let K be an arbitrary d-dimensional maximal S-free set. It suffices to show that K is contained in an S-free polyhedron with at most h facets. We consider a sequence $(P_n)_{n=1}^{+\infty}$ of d-dimensional rational polytopes such that

$$P_n \subseteq P_{n+1} \quad \forall n \in \mathbb{N} \tag{8}$$

and

$$\operatorname{int}(K) = \bigcup_{n=1}^{+\infty} P_i. \tag{9}$$

Such polytopes P_n can be constructed as follows. Let $(z_n)_{n=1}^{+\infty}$ be a sequence of all rational points of $\operatorname{int}(K)$ such that the first $d+1$ points $z_1, \ldots, z_{d+1}$ are affinely independent. Then, for every $n \in \mathbb{N}$, we define P_n to be the convex hull of $\{z_1, \ldots, z_{n+d}\}$. By the assumption, each P_n is contained in an S-free polyhedron Q_n having at most f facets. Every Q_n can be represented by

$$Q_n = \Big\{ x \in \mathbb{R}^d \, : \, \langle u_{1,n} \, , \, x \rangle \le \beta_{1,n}, \ldots, \langle u_{f,n} \, , \, x \rangle \le \beta_{f,n} \Big\}$$

where $u_{1,n}, \dots, u_{f,n} \in \mathbb{R}^d$ are vectors of unit (Euclidean) length and $\beta_{1,n}, \dots, \beta_{f,n} \in \mathbb{R}$. There exists an infinite subset $\mathbb{N}_\infty$ of $\mathbb{N}$ such that, for every $i \in [f]$, the vector $u_{i,n}$ converges to some unit vector u_i and β_i converges to some $\beta_i \in (-\infty, +\infty]$, as n goes to infinity over points of $\mathbb{N}_\infty$. We define the polyhedron

$$Q := \left\{ x \in \mathbb{R}^d \, : \, \langle u_1 \, , \, x \rangle \le \beta_1, \dots, \langle u_f \, , \, x \rangle \le \beta_f \right\}.$$

By construction, $P_1 \subseteq P_n \subseteq Q_n$ for every $n \in \mathbb{N}$. Hence $P_1 \subseteq Q$, which shows that Q is d-dimensional. Let us show that Q is S-free. We assume the contrary. Then there exists $x \in S$ belonging to $\operatorname{int}(Q) = \left\{ x \in \mathbb{R}^d \, : \, \langle u_1 \, , \, x \rangle < \beta_1, \dots, \langle u_f \, , \, x \rangle < \beta_f \right\}$. The latter implies $\langle u_{i,n} \, , \, x \rangle < \beta_{i,n}$ for all $i \in [f]$ if $n \in \mathbb{N}_\infty$ is sufficiently large. This implies $x \in S \cap \operatorname{int}(Q_n)$ for all sufficiently large $n \in \mathbb{N}_\infty$, contradicting the fact that Q_n is S-free. We also show $\operatorname{int}(K) \subseteq Q$ arguing by contradiction. If x is a point belonging to $\operatorname{int}(K)$ but not to Q, then one can fix $i \in [f]$ such that $\langle u_i \, , \, x \rangle > \beta_i$. Consequently, $\langle u_{i,n} \, , \, x \rangle > \beta_{i,n}$ for all sufficiently large $n \in \mathbb{N}_\infty$. The inequality $\langle u_{i,n} \, , \, x \rangle > \beta_{i,n}$ implies $x \notin Q_n$ and, by this, $x \notin P_n$. Thus, $x \notin P_n$ for all sufficiently large $n \in \mathbb{N}_\infty$. Since the sequence of P_n's is monotone (as described by (8)), we get $x \notin P_n$ for every $n \in \mathbb{N}$. Consequently, $x \notin \bigcup_{n=1}^{+\infty} P_n$. In view of (9), we obtain $x \notin \operatorname{int}(K)$, which is a contradiction. We have verified the inclusion $\operatorname{int}(K) \subseteq Q$. Taking the closure of the left and the right hand side we arrive at $K \subseteq Q$. This finishes the proof. $\square$

Proof of Theorem 4. Let us first consider degenerate cases. If $S = \emptyset$, we have $h(S) = h(\{\emptyset\}) = 0$. On the other hand, for $S = \emptyset$, the whole space $\mathbb{R}^d$ is the only maximal S-free set, and thus $f(S) = 0$. If S is nonempty we have $h(S) \in \mathbb{N}$ or $h(S) = +\infty$. In the case $h(S) = +\infty$ the assertion is trivial. Now assume $h(S) \in \mathbb{N}$.

Let us verify the assumption of Lemma 6 for $f := h(S)$. Let P be an arbitrary d-dimensional S-free rational polyhedron in $\mathbb{R}^d$. We represent P by $P = H_1 \cap \cdots \cap H_n$, where $n \in \mathbb{N}$ and $H_1, \dots, H_n$ are closed rational halfspaces. Then $\operatorname{int}(H_1) \cap S, \dots, \operatorname{int}(H_n) \cap S$ are S-convex sets whose intersection is empty. By the definition of the Helly number $h(S)$, there exist indices $i_1, \dots, i_f \in [n]$ such that $(\operatorname{int}(H_{i_1}) \cap \cdots \cap \operatorname{int}(H_{i_f})) \cap S = \emptyset$. It follows that $P \subseteq Q := H_{i_1} \cap \cdots \cap H_{i_f}$, where Q is an S-free polyhedron with at most f facets. Thus, the assumption of Lemma 6 is fulfilled. Lemma 6 yields the assertion. $\square$

Proof of Theorem 5. Directly from the definition of the Helly number it follows that for every $S \subseteq \mathbb{R}^d$ and every convex set $A \subseteq \mathbb{R}^d$ one has

$$h(S \cap A) \le h(S). \tag{10}$$

Using Theorem 4, (10) and Doignon's theorem (represented by (2)) we obtain $f(A \cap \mathbb{Z}^d) \le h(A \cap \mathbb{Z}^d) \le h(\mathbb{Z}^d) = 2^d$, which shows (4).

For the verification of (5) it suffices to establish the existence of maximal $\mathbb{Z}^d$-free polyhedra with 2^d facets. Such polyhedra can easily be constructed. Let $\| \cdot \|_1$ be the l_1-norm in $\mathbb{R}^d$ and let c be the vector in $\mathbb{R}^d$ whose components are all equal to $1/2$. Let $P := \left\{ x \in \mathbb{R}^d \, : \, \|x - c\|_1 \le d/2 \right\}$. The polytope P is $\mathbb{Z}^d$-free since $\|z - c\|_1 \ge d/2$ for all $z \in \mathbb{Z}^d$ and is maximal $\mathbb{Z}^d$-free since each of the 2^d facets of P is 'blocked' by a point from $\{0,1\}^d$. This shows $f(\mathbb{Z}^d) \ge 2^d$ and yields (5).

For every $S \subseteq \mathbb{R}^d$ the trivial equality $f(S \times \mathbb{R}^n) = f(S)$ holds. The latter equality and Doignon's theorem yield $f(\mathbb{Z}^d \times \mathbb{R}^n) = f(\mathbb{Z}^d) = 2^d$, which verifies (7). Inequality (4) is a straightforward consequence of (7) and (10). $\square$

Remark 7. As can be seen from the proof of Theorem 4, the inequality $f(S) \le h(S)$ can be improved to

$$f(S) \le h\left(\left\{\operatorname{int}(P) \cap S \,:\, P \text{ is a rational polyhedron in } \mathbb{R}^d\right\}\right).$$

Thus, in the proof of Theorem 5 it is sufficient to apply the following 'rational' version of Helly's theorem:

$$h\left(\left\{\operatorname{int}(P) \cap S \,:\, P \text{ is a rational polyhedron in } \mathbb{R}^d\right\}\right) = 2^d. \tag{11}$$

Note that, for (11), replacing $\operatorname{int}(P)$ by P does not change the family on the left hand side. A very short proof of (11) was given by Bell [6].

We also remark that (11) can be used to show $h(\mathbb{Z}^d) = 2^d$ (the full version of Doignon's theorem). This is done as follows. Let $n \in \mathbb{N}$, $n \ge 2^d$ and let $A_1, \ldots, A_n$ be convex sets in $\mathbb{R}^d$ such that for all $1 \le i_1 < \cdots < i_{2^d} \le n$ one has $A_{i_1} \cap \cdots \cap A_{i_{2^d}} \cap \mathbb{Z}^d \neq \emptyset$. Using (11) we now show $A_1 \cap \cdots \cap A_n \cap \mathbb{Z}^d \neq \emptyset$. Consider the box $B := [-N, N]^d$ with $N > 0$ large enough to guarantee that for all $1 \le i_1 < \cdots < i_{2^d} \le n$ one has $A_{i_1} \cap \cdots \cap A_{i_{2^d}} \cap B \cap \mathbb{Z}^d \neq \emptyset$. Since, for every $i \in [n]$, the convex hull of $A_i \cap B \cap \mathbb{Z}^d$ ($i \in [n]$) is an integral polytope, one can determine rational polytopes $P_1, \ldots, P_n$ such that, for every $i \in [n]$, one has $A_i \cap B \cap \mathbb{Z}^d = \operatorname{int}(P_i) \cap \mathbb{Z}^d$. Applying (11) to the sets $\operatorname{int}(P_1) \cap \mathbb{Z}^d, \ldots, \operatorname{int}(P_n) \cap \mathbb{Z}^d$ we deduce $\operatorname{int}(P_1) \cap \cdots \cap \operatorname{int}(P_n) \cap \mathbb{Z}^d \neq \emptyset$. Hence $A_1 \cap \cdots \cap A_n \cap \mathbb{Z}^d \neq \emptyset$. This implies $h(\mathbb{Z}^d) \le 2^d$. The inequality $h(\mathbb{Z}^d) \ge 2^d$ follows by considering the family $\mathcal{F}$ of 2^d sets $\{0,1\}^d \setminus \{z\}$ with $z \in \{0,1\}^d$. The elements of $\mathcal{F}$ are $\mathbb{Z}^d$-convex, the intersection of $\mathcal{F}$ is emtpy, and the intersection of every nonempty proper subfamily of $\mathcal{F}$ is nonempty.

Remark 8. We indicate that the authors of [7] work under somewhat more general assumptions than the assumptions of Theorem 1. They consider the following sets:

- an affine subspace W of $\mathbb{R}^n$ (where $n \in \mathbb{N}$);
- a subset S of $W \subseteq \mathbb{Z}^n$ such that $S = \mathbb{Z}^d \cap C$ for some convex set $C \subseteq W$;
- a closed, convex set K such that $K \subseteq W$, the relative interior of K does not contain points of S and such that K is inclusion-maximal with respect to the above properties.

Also in this more general situation the polyhedrality of K and a bound on the number of facets of K can be determined using Theorem 4 and Doignon's theorem. Let d be the dimension of W. It suffices to consider the case that K is d-dimensional and $S \neq \emptyset$. Let us fix a nonsingular affine transformation $T : W \to \mathbb{R}^d$ which maps some point of $W \cap \mathbb{Z}^d$ to the origin. Then $\Lambda := T(W \cap \mathbb{Z}^d)$ is a lattice in $\mathbb{R}^d$ and $T(K)$ is a maximal $T(S)$-free set. Doignon's theorem implies $h(\Lambda) = 2^r$, where r is the rank of Λ. Furthermore, $h(T(S) \cap \Lambda) \le h(\Lambda)$. Taking into account Theorem 4 we deduce that $T(K)$ (and, by this, also K) is a polyhedron with at most 2^r facets.

References

[1] K. Andersen, Q. Louveaux, R. Weismantel, and L. A. Wolsey, *Inequalities from two rows of a simplex tableau*, Integer programming and combinatorial optimization, Lecture Notes in Comput. Sci., vol. 4513, Springer, Berlin, 2007, pp. 1–15.

[2] G. Averkov and R. Weismantel, *Transversal numbers over subsets of linear spaces*, 10pp., Preprint arXiv:1002.0948 (to appear in *Adv. Geom.*), 2009.

[3] E. Balas, *Intersection cuts—a new type of cutting planes for integer programming*, Operations Res. **19** (1971), 19–39.

[4] A. Basu, M. Conforti, G. Cornuéjols, and G. Zambelli, *Maximal lattice-free convex sets in linear subspaces*, Math. Oper. Res. **35** (2010), no. 3, 704–720.

[5] ______, *Minimal inequalities for an infinite relaxation of integer programs*, SIAM J. Discrete Math. **24** (2010), no. 1, 158–168.

[6] D. E. Bell, *A theorem concerning the integer lattice*, Studies in Appl. Math. **56** (1977), no. 2, 187–188.

[7] S. S. Dey and D. A. Morán R., *On maximal S-free convex sets*, SIAM J. Discrete Math. **25** (2011), no. 1, 379–393.

[8] S. S. Dey and L. A. Wolsey, *Constrained infinite group relaxations of MIPs*, SIAM J. Optim. **20** (2010), no. 6, 2890–2912.

[9] J.-P. Doignon, *Convexity in crystallographical lattices*, J. Geometry **3** (1973), 71–85.

[10] R. Fukasawa and O. Günlük, *Strengthening lattice-free cuts using non-negativity*, Discrete Optim. **8** (2011), no. 2, 229–245.

[11] L. Lovász, *Geometry of numbers and integer programming*, Mathematical programming (Tokyo, 1988), Math. Appl. (Japanese Ser.), vol. 6, SCIPRESS, Tokyo, 1989, pp. 177–201.

[12] H. E. Scarf, *An observation on the structure of production sets with indivisibilities*, Proc. Nat. Acad. Sci. U.S.A. **74** (1977), no. 9, 3637–3641.

[13] R. Schneider, *Convex Bodies: The Brunn-Minkowski Theory*, Encyclopedia of Mathematics and its Applications, vol. 44, Cambridge University Press, Cambridge, 1993.

[14] A. Schrijver, *Theory of Linear and Integer Programming*, Wiley-Interscience Series in Discrete Mathematics, John Wiley & Sons Ltd., Chichester, 1986, A Wiley-Interscience Publication.

[15] M. J. Todd, *The number of necessary constraints in an integer program: A new proof of Scarf's theorem*, Technical Report 355, School of Operations Research and Industrial Engineering, Cornell University, Ithaca, N.Y., 1977.